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Algebraic Structure of Quantum Fluctuations

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On the basis of the existence of second and third moments of fluctuations, we prove a theorem about the Lie-algebraic structure of fluctuation operators. This result gives insight into the quantum character of fluctuations. We illustrate the presence of a Lie algebra of fluctuation operators in a model of the anharmonic crystal, and show the dependence of the Lie-algebra structure on the fine structure of the fluctuation operator algebra. The result is also applied to construct the normal Goldstone mode in the ideal Bose gas for Bose–Einstein condensation.

KEY WORDS: Quantum fluctuations; Lie-algebraic structure; extremal states; Goldstone theorem.

1. INTRODUCTION

In refs. [1, 2], a theory has been developed in order to establish the quantum structure of *normal fluctuations* for quantum lattice systems. One considers a cubic lattice $\mathbb{Z}^\nu$. At each lattice site $x \in \mathbb{Z}^\nu$ one associates a C^*-algebra $\mathscr{A}_x$ of single-site observables. The observables measurable within the volume Λ is given by the algebra $\mathscr{A}_\Lambda = \bigotimes_{x \in \Lambda} \mathscr{A}_x$. The typical examples are quantum spin models where $\mathscr{A}_x$ is a matrix algebra. The algebra of observables for the infinite system is given by the C^*-inductive limit $\mathscr{A}$ of the minimal tensor product algebras $\{\mathscr{A}_\Lambda, \Lambda \subset \mathbb{Z}^\nu\}$. Denote by τ_x the space translation $*$-automorphism of $\mathscr{A}$. Typically one considers a set of local states ω_Λ for each volume, in particular Gibbs states for some local Hamiltonian or a ground state, such that the weak limit of the ω_Λ, $\Lambda \to \mathbb{Z}^\nu$, tends to an

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extremal translation invariant state ω, i.e. $\omega \cdot \tau_x = \omega$ and for all $A, B \in \mathscr{A}$, $\lim_{|x| \to \infty} \omega(A \tau_x B) = \omega(A)\, \omega(B)$.

If the functions $x \to \omega(A\tau_x B) - \omega(A)\,\omega(B)$ are $\ell^1(\mathbb{Z}^\nu)$-functions, one proves that the limits

$$F^0(A) \equiv \lim_{\Lambda \to \mathbb{Z}^\nu} F_0^\Lambda(A) \tag{1.1}$$

of the local fluctuations

$$F_\Lambda^0(A) = \frac{1}{\sqrt{|\Lambda|}} \sum_{x \in \Lambda} (\tau_x A - \omega_\Lambda(A)) \tag{1.2}$$

exist in the sense of a non-commutative central limit theorem.[1,2] Moreover one shows that the fluctuations $\{F^0(A), A \in \mathscr{A}\}$ are given by a representation of the canonical commutation relations, satisfying

$$[F^0(A), F^0(B)] = \lim_\Lambda \omega_\Lambda([F_\Lambda^0(A), F_\Lambda^0(B)]) \tag{1.3}$$

This formula marks the non-commutative character of the fluctuations.

This representation is determined as the GNS-representation of a quasi-free state $\tilde{\omega}$ on the Bose field algebra generated by the fluctuations $\{F^0(A) \mid A \in \mathscr{A}\}$. The state $\tilde{\omega}$ is determined by the formula

$$\tilde{\omega}(e^{i\lambda F^0(A)}) = \lim_\Lambda \omega_\Lambda(e^{i\lambda F_\Lambda^0(A)}) = \exp\left(-\frac{\lambda^2}{2} \sum_x (\omega(A\tau_x A) - \omega(A)^2)\right) \tag{1.4}$$

The ℓ^1-cluster property ensures the Gaussian character of the limit and the convergence of the exponential in formula (1.4). It is important to realize that all these results apply only to situations where the fluctuations are normal (ℓ^1-cluster property), but the case of long range correlation systems at phase transitions, do not fit into the schema above. A first attempt to generalize the theory to this more interesting situation has been given in ref. [3]. There is proved that the abnormal fluctuations generate a Lie algebra which can be more general than the Heisenberg-algebra. An example of this more general situation is already given by the harmonic crystal in the ground state.[3]

It is the aim of this paper to give a proof of the Lie algebra character of the fluctuations (for definitions see Section 2) solely under the condition of the existence of the variance (second moment) and the third moment. It means that the Lie algebra structure emerges already without assuming the existence of all moments, or the characteristic function. The setting of our proof (Section 3) amounts to prove an Inönü–Wigner contraction theorem,[4,5] which characterizes the set of abnormal fluctuations as an abstract Lie algebra, under conditions which are much weaker than in ref. [3]. The price to be paid is that we cannot identify the representations of the Lie algebra. Nevertheless, the step forward is that we show that the second and third moment already determines the Lie algebra or non-commutative structure of the algebra of abnormal fluctuations. This theorem gives detailed information about the quantum character of quantum systems at its level of fluctuations. Our theorem is illustrated in the models of the anharmonic and harmonic crystals (Section 4). In ref. [3], the first illustration was given for the toy model of the harmonic crystal. Here we generalize this result to a more realistic model, extending considerably the result on the CCR-algebra given in ref. [6].

Our main theorem is also used in the explicit and rigorous construction of the normal mode canonical variables associated with the Goldstone Boson, which appears as a consequence of spontaneous gauge symmetry breaking in the Boson state with condensation (Section 5). It is the first mathematically rigorous construction of the long wavelength, low frequency normal mode in the presence of spontaneous symmetry breaking. Be it done here only for the ideal Bose gas, we strongly believe that this is a first step towards a general construction.

2. FLUCTUATIONS

An abstract *Lie algebra* is an n-dimensional vector space $\mathscr{G}$ with basis $\{v_i\}_{i=1,\dots,n}$ and with product

$$v_j \cdot v_k \equiv [v_j, v_k] = \sum_{\ell=1}^{n} c_{jk}^{\ell} v_{\ell} \tag{2.1}$$

where the c_{jk}^{ℓ} are the structure constants for that basis satisfying:

$$c_{jk}^{\ell} + c_{kj}^{\ell} = 0 \tag{2.2}$$

$$\sum_r (c_{ij}^r c_{rk}^s + c_{jk}^r c_{ri}^s + c_{ki}^r c_{rj}^s) = 0 \qquad \text{(Jacobi identity)} \tag{2.3}$$

Consider here a concrete Lie algebra $\mathscr{G}$ of operators in $\mathscr{A}_{\{0\}}$.

Let

$$\{L_0 = i1, L_1, \dots, L_m\}; \qquad m < \infty \tag{2.4}$$

be the basis of the Lie algebra $\mathscr{G}$, such that

$$L_j^* = -L_j; \qquad j = 0, 1, \dots, m$$

$\omega(L_j) \equiv \lim_{\Lambda} \omega_{\Lambda}(L_j) = 0$ for $j > 0$.

Clearly, $\omega_\Lambda(L_0) = i$ for all Λ. The Lie algebra $\mathscr{G}$ is always non-semisimple. For the basis $\{L_j\}$, we have the product rule

$$[L_j, L_k] = \sum_{\ell=0}^{m} c_{jk}^{\ell} L_\ell \tag{2.5}$$

with real structure constants. Because of the special choice of $L_0 = i1$, one has $c_{0k}^{\ell} = c_{k0}^{\ell} = 0$ and because the means of the $(L_j)_{j=1,\dots,m}$ vanish, one has $c_{jk}^0 = -i \lim_\Lambda \omega_\Lambda([L_j, L_k])$. We consider now the fluctuations of the generators of $\mathscr{G}$ and are looking for a characterization of the Lie algebra of these fluctuations if any.

For each finite volume $\Lambda \subset \mathbb{Z}^\nu$, define the local fluctuations

$$L_j(\Lambda) = \frac{1}{|\Lambda|^{1/2+\delta_j}} \sum_{x \in \Lambda} [\tau_x L_j - \omega_\Lambda(L_j)]; \qquad j = 1,\dots, m \tag{2.6}$$

$$L_0(\Lambda) = i\mathbb{1} = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} i1 = i1 \tag{2.7}$$

For $j = 1,\dots, m$, $L_j(\Lambda)$ is the local fluctuation around its equilibrium value. The unit operator (2.7) does not fluctuate. However for notational convenience, we introduce (2.7) nevertheless. Now we formulate the basic conditions for our purposes.

Condition A. We assume that the parameters δ_j are determined by the existence of the finite and non trivial variances:

$$0 < -\lim_\Lambda \omega_\Lambda(L_j(\Lambda)^2) < \infty; \qquad j = 0, 1,\dots, m \quad \square \tag{2.8}$$

Remark that the parameter δ_0 in (2.7), is already been taken as $\delta_0 = 1/2$, i.e. the condition (2.8) is satisfied for $j = 0$. After reordering the basis of $\mathscr{G}$ one can put:

$$\delta_0 = \tfrac{1}{2} > \delta_1 \geqslant \delta_2 \geqslant \cdots \geqslant \delta_m > -\tfrac{1}{2} \tag{2.9}$$

Condition B. We assume that all third moments are finite, i.e. for $j, k, \ell = 1,\dots, m$ one has:

$$\lim_\Lambda |\omega_\Lambda(L_j(\Lambda)\, L_k(\Lambda)\, L_\ell(\Lambda))| < \infty \quad \square \tag{2.10}$$

Remarks. Although, we are not going here into the details of the precise form of the sequence $\{\omega_\Lambda\}_\Lambda$ of states, we have in mind, that ω_Λ is a Gibbs state for some local Hamiltonian H_Λ i.e. for all $A \in \mathscr{A}$; $\omega_\Lambda(A) = \operatorname{tr} e^{-\beta H_\Lambda} A / \operatorname{tr} e^{-\beta H_\Lambda}$. The limit $\Lambda \to \mathbb{Z}^\nu$ is in the sense of increasing cubes $\Lambda = [-L, L]^\nu$, and (2.8) can depend very strongly on the boundary conditions taken for defining H_Λ (see e.g. ref. [6] and Section 4).

The parameters δ_j are determined by the conditions (2.8) for the existence and non-triviality of the variances. If for some $j \geqslant 1$, the corresponding $\delta_j = 0$, then the operator L_j has a *normal fluctuation operator* $\mathscr{L}_j = \lim_\Lambda L_j(\Lambda)$, where the limit is understood in the sense of (2.8). It is proved in ref. [2], that one has normal gaussian fluctuations if the function

$$x \in \mathbb{Z}^\nu \to \omega(L_j \tau_x L_j), \qquad j \geqslant 1$$

is an $\ell^1(\mathbb{Z}^\nu)$-function. On the other hand, if one has long-range correlations, e.g. if

$$\omega(L_j \tau_x L_j) \simeq 0 \left(\frac{1}{|x|^{\nu - 2 + \eta_j}} \right)$$

with $\eta_j < 2$, then the parameter δ_j is related to the η_j, the critical exponent of the static susceptibility for L_j, by the relation:

$$\eta_j = 2(1 - \nu \delta_j)$$

This relation leads to $0 < \nu - 2 + \eta_j = \nu(1 - 2\delta_j)$ or $\delta_j < \frac{1}{2}$. The indices δ are a measure to detect at which level of space scaling, fluctuations do become visible. Because of the above argument, we limit our discussion to values $\delta_j < \frac{1}{2}$ for $1 \leqslant j \leqslant m$ (see (2.9)). Also $\delta = \frac{1}{2}$ yields the law of large numbers, i.e. averages and not fluctuations. Remark that we put formally $\delta_0 = \frac{1}{2}$.

Furthermore, in order to satisfy (2.8), it can happen that the parameter δ_j has to be chosen negative. An explicit example is given in ref. [6], in the case that ω is a ground state on the critical line. In this case, it is reasonable to limit our discussion to the situation that all $\delta_j > -\frac{1}{2}$ (see (2.9)). In all cases, if $\delta_j \neq 0$, then the fluctuation

$$\mathscr{L}_j = \lim_\Lambda L_j(\Lambda)$$

is called an *abnormal fluctuation*.

Finally we would like to stress that Condition B (2.10) about the finiteness of the third moments is independent of Condition A (2.8), because new cluster properties on the correlation function are implied.

Lemma 2.1. If Condition A is satisfied, the limit set $\{\mathscr{L}_j\}_{j=0,1,\dots,m}$ generates a pre-Hilbert space $\mathscr{H}_0$, with scalar product

$$(\mathscr{L}_j, \mathscr{L}_k) = \lim_{\Lambda} \omega_\Lambda(L_j^*(\Lambda)\, L_k(\Lambda)) \tag{2.11}$$

and $\dim \mathscr{H}_0 \leqslant m+1$.

Proof. From (2.8) and Schwartz-inequality, one gets:

$$|\omega_\Lambda(L_j^*(\Lambda)\, L_k(\Lambda))|^2 \leqslant \omega_\Lambda(L_j(\Lambda)^2)\, \omega_\Lambda(L_k(\Lambda)^2)$$

By the compactness argument the sequence (2.11) has an accumulation point. By polarization and (2.8) the limit exists. □

Suppose that it happens that $\delta_j = \delta_{j+1}$, i.e. the fluctuations of L_j and of L_{j+1} do appear at the same space scaling level, then there are two possibilities. Either, $\mathscr{L}_j$ and $\mathscr{L}_{j+1}$ are linearly dependent in $\mathscr{H}_0$, then we identify them (we consider the corresponding equivalence classes in $\mathscr{H}_0$), or $\mathscr{L}_j$ and $\mathscr{L}_{j+1}$ are independent. In the latter case, we rearrange the basis $\{L_j\}_j$ of $\mathscr{G}$ (Gram–Schmidt orthogonalization) such that:

$$(\mathscr{L}_j, \mathscr{L}_{j+1}) = 0 \tag{2.12}$$

As a result, Lemma 2.1, defines the limits $(\mathscr{L}_j)_{j=0,1,\dots,m}$ as elements of a pre-Hilbert space.

3. LIE ALGEBRA OF FLUCTUATIONS

We consider now the Lie product of the fluctuations of the generators $\{L_j\}_{j=0,1,\dots,m}$ of the algebra $\mathscr{G}$ (2.4)–(2.5), given for each finite volume Λ, by the commutator in $\mathscr{A}$:

$$[L_j(\Lambda), L_k(\Lambda)] = \frac{1}{|\Lambda|^{1+\delta_j+\delta_k}} \sum_{x,\, y \in \Lambda} [\tau_x L_j, \tau_y L_k] = \sum_{\ell=0}^{m} c_{jk}^{\ell}(\Lambda)\, L_\ell(\Lambda) \tag{3.1}$$

where we used the locality in $\mathscr{A}$, (2.5) and the notations:

$$\left.\begin{aligned} c_{jk}^{\ell}(\Lambda) &= \frac{c_{jk}^{\ell}}{|\Lambda|^{1/2+\delta_j+\delta_k-\delta_\ell}} \quad \ell = 1,\dots, m \\ c_{jk}^{0}(\Lambda) &= -i\omega_\Lambda([L_j(\Lambda), L_k(\Lambda)]) \end{aligned}\right\} \tag{3.2}$$

In particular one has

$$c^0_{jk}(\Lambda) = |\Lambda|^{-\delta_j - \delta_k} \sum_{\ell=0}^{m} c^{\ell}_{jk} \omega_\Lambda(L_\ell(\Lambda)) \tag{3.3}$$

If moreover *Condition C*, namely that $\lim_\Lambda c^0_{jk}$ (3.3) exists, then

Lemma 3.1. For each finite volume, the set $\{L_j(\Lambda)\}_{j=0,1,\ldots,m}$ generates a Lie algebra $\mathscr{G}(\Lambda)$ of dimension $m+1$ and with structure constants the $c^{\ell}_{jk}(\Lambda)$, given by (3.2).

Proof. It is an easy exercise to check that the constants $c^{\ell}_{jk}(\Lambda)$, defined by (3.2) satisfy, the symmetry property (2.2):

$$c^{\ell}_{jk}(\Lambda) + c^{\ell}_{kj}(\Lambda) = c^{\ell}_{jk} + c^{\ell}_{kj} = 0$$

and the Jacobi identity (2.3):

$$\sum_r (c^r_{ij}(\Lambda)\, c^s_{rk}(\Lambda) + c^r_{jk}(\Lambda)\, c^s_{ri}(\Lambda) + c^r_{ki}(\Lambda)\, c^s_{rj}(\Lambda)) = 0$$

proving the lemma. □

Clearly by considering local fluctuations one constructs a map from the Lie algebra $\mathscr{G}$ onto the Lie algebra $\mathscr{G}(\Lambda)$, by a non-trival change of structure constants given by (3.2). This change is not just a change of basis of the Lie algebra, it becomes singular in the limit Λ tending to infinity. When the transformed structure constants approach a well defined limit, and the transformation becomes singular, a new, non-isomorphic Lie algebra might appear. The limit algebra $\mathscr{G}(\mathbb{Z}^\nu)$, called the contracted one of the original, is always non-semisimple. This contraction is a typical Inönü–Wigner contraction.[4,5] Our main theorem yields a proof of the existence of the algebra $\mathscr{G}(\mathbb{Z}^\nu)$, by proving the limit of the structure constants (3.2).

Theorem 3.2. If the conditions A and B are satisfied, then

(i) $\lim_\Lambda c^{\ell}_{jk}(\Lambda) = 0, \quad$ if $\frac{1}{2} + \delta_j + \delta_k - \delta_\ell > 0$

(ii) $\lim_\Lambda c^{\ell}_{jk}(\Lambda) = c^{\ell}_{jk}, \quad$ if $\frac{1}{2} + \delta_j + \delta_k - \delta_\ell = 0$

(iii) $c^{\ell}_{jk} = 0, \quad$ if $\frac{1}{2} + \delta_j + \delta_k - \delta_\ell < 0$

Proof. The cases (i) and (ii) are trivial. We prove (iii). First we prove that:

$$\delta_j + \delta_k < 0 \qquad \text{holds only if} \quad c^0_{jk} = 0 \tag{α}$$

By simple computation, using $\omega(L_\ell(\Lambda))=0$ for $\ell \neq 0$, $\delta_0=\frac{1}{2}$, $\omega(L_0(\Lambda))=i$ and (2.9), one gets:

$$\lim_\Lambda |\omega_\Lambda([L_j(\Lambda), L_k(\Lambda)])|\ |\Lambda|^{\delta_j+\delta_k} = \lim_\Lambda |c^0_{jk}(\Lambda)|\ |\Lambda|^{\delta_j+\delta_k}$$

$$= \lim_\Lambda \left| \sum_{\ell'=0}^{m} \frac{c^{\ell'}_{jk}}{|\Lambda|^{1/2-\delta'}} \omega_\Lambda(L_{\ell'}(\Lambda)) \right| = |c^0_{jk}|$$

But by Schwartz inequality, Condition A (2.8) and the fact that $\delta_j+\delta_k<0$:

$$|c^0_{jk}| = \lim_\Lambda |\Lambda|^{\delta_j+\delta_k} |\omega_\Lambda([L_j(\Lambda), L_k(\Lambda)])|$$

$$\leqslant 2 \lim_\Lambda |\Lambda|^{\delta_j+\delta_k} [\omega_\Lambda(L_j(\Lambda)^2)\, \omega_\Lambda(L_k(\Lambda)^2)]^{1/2} = 0$$

proving (α).

Now we prove that

$$\tfrac{1}{2}+\delta_j+\delta_k-\delta_\ell<0 \qquad \text{holds only if} \quad c^\ell_{jk}=0 \tag{β}$$

We prove it by induction on the index ℓ. For the induction case take $\ell=0$, then

$$0>\tfrac{1}{2}+\delta_j+\delta_k-\delta_0=\delta_j+\delta_k$$

In (α) we proved that then $c^0_{jk}=0$. Hence $\lim_\Lambda c^0_{jk}(\Lambda)=c^0_{jk}=0$.

By induction, suppose that the statement (β) holds for all $\ell'<\ell$, and that $\frac{1}{2}+\delta_j+\delta_k-\delta_\ell<0$. This implies that, because of (2.9) also $\frac{1}{2}+\delta_j+\delta_k-\delta_{\ell'}<0$.

Using the induction hypothesis, consider now:

$$\lim_\Lambda |\omega_\Lambda([L_j(\Lambda), L_k(\Lambda)]L_\ell(\Lambda))|\ |\Lambda|^{1/2+\delta_j+\delta_k-\delta_\ell}$$

$$= \lim_\Lambda \left| \sum_{\ell'=0}^{m} c^{\ell'}_{jk} |\Lambda|^{\delta_{\ell'}-\delta_\ell} \omega_\Lambda(L_{\ell'}(\Lambda)\, L_\ell(\Lambda)) \right|$$

$$= \lim_\Lambda \left| \sum_{\ell'=\ell}^{m} c^{\ell'}_{jk} |\Lambda|^{\delta_{\ell'}-\delta_\ell} \omega_\Lambda(L_{\ell'}(\Lambda)\, L_\ell(\Lambda)) \right|$$

$$= \lim_\Lambda |c^\ell_{jk}|\ |\omega_\Lambda(L_\ell(\Lambda)^2)|$$

The last equality is obtained as follows. If $\ell'>\ell$ but $\delta_{\ell'}=\delta_\ell$, then one uses (2.12), if $\ell'>\ell$ and $\delta_{\ell'}-\delta_\ell<0$ (2.9), then it follows that in the sum over $\ell'\geqslant\ell$, only the term $\ell'=\ell$ remains. Now, using Condition A (2.8) and B (2.10), together with the fact that $\frac{1}{2}+\delta_j+\delta_k-\delta_\ell<0$, one gets $c^\ell_{jk}=0$. This proves (β) and (iii). $\square$

This theorem settles the limit Lie algebraic structure of the fluctuations algebra $\mathscr{G}(\mathbb{Z}^\nu)$ as an abstract Lie algebra with generators $\{\mathscr{L}_i\}_{i=0,1,...,m}$ and structure constants $(\lim_\Lambda c_{jk}^{\ell}(\Lambda))$.

It is interesting to distinguish a number of special cases:

(a) for normal fluctuations all $\delta_j=0$, except $\delta_0=\frac{1}{2}$, hence from the theorem one gets

$$\lim_\Lambda c_{jk}^{\ell}(\Lambda)=0 \qquad \text{for} \quad \ell=1,2,...,m$$

$$\lim_\Lambda c_{jk}^{0}(\Lambda)=c_{jk}^{0}=-i\omega([L_j,L_k])$$

This means that the $\mathscr{L}_j$ are the generators of a Lie algebra such that

$$[\mathscr{L}_j,\mathscr{L}_k]=c_{jk}^{0}=i\sigma_\omega(L_j,L_k) \tag{3.4}$$

where

$$\sigma_\omega(L_j,L_k)=-\omega([L_j,L_k])$$

and σ_ω is a symplectic form on the initial algebra $\mathscr{G}$. The $\{\mathscr{L}_j\}$ form a Boson field satisfying the canonical commutation relations (CCR) (3.4).

(b) If $\frac{1}{2}+\delta_j+\delta_k-\delta_\ell>0$ for all j,k,ℓ, one obtains a commutative algebra of fluctuations.

(c) One has the richest structure if $\frac{1}{2}+\delta_j+\delta_k-\delta_\ell=0$ for all j,k,ℓ or for some of them. We obtain a phenomenon of *scale invariance* of the physical system $(\mathscr{G},\omega)$ i.e. the $c_{jk}^{\ell}(\Lambda)$ are independent of the volume Λ (3.2), then algebras different from the CCR-algebra are observed (see e.g. ref. [3] for an example).

One recovers again the CCR-structure although the fluctuations are abnormal if $\delta_j=-\delta_k\neq 0$, i.e. one of the parameters is negative e.g. $\delta_j<0$ and the operator $\mathscr{L}_j$ is the result of space squeezing, $\delta_k>0$ and $\mathscr{L}_k$ is dilated. This yields a microscopic explanation of the phenomenon of squeezing in quantum optics (see e.g. ref. [6]). A new application of this situation is given in the next section.

4. QUANTUM FLUCTUATIONS IN ORDERED AND DISORDERED STATES

As the first application of the above general theorem we discuss the Lie structure of the algebra of fluctuation operators in a model of quantum ferroelectrics[6] in the ordered phase, as well as in a toy harmonic crystal model,[3] where evidently there is no ordering.

It is in the quantum ferroelectric model that the nontrivial quantum character of critical fluctuations was observed for the first time: fluctuations generated by the canonical operators of displacements $\{Q_\ell\}_{\ell \in \mathbb{Z}^d}$, $Q_\ell \in \mathbb{R}^1$, and momenta $\{P_\ell\}_{\ell \in \mathbb{Z}^d}$ become non-abelian on the critical line when the critical temperature $T_c(\lambda) \to 0$ for $\lambda \to \lambda_c$. Here $\lambda = \hbar/\sqrt{m}$ is the quantum parameter of the model, where m is the mass of the atoms in the sites of a ferroelectric lattice.

We shall show that this non-abelian property at $T_c(\lambda_c) = 0$ is not by accident. As was observed in ref. [6] there is no ferroelectric order for any temperature if $\lambda \geqslant \lambda_c$ (for the general case see ref. [7]). On the other hand in ref. [6] it was also discovered that below the critical line, i.e., for $\lambda < \lambda_c$ and $T < T_c(\lambda)$ (ordered phase domain), fluctuations get sensitive to the quasi-average type of perturbation even in the pure phases. In this section we extend the result of refs. [6, 8] and compute the critical exponents δ_Q, δ_P on the line $\lambda \in (0, \lambda_c)$ at $T = 0$.

Moreover the Lie structure has been already observed in the framework of a toy harmonic model.[3] In spite of the similarity between the harmonic model and anharmonic one in ref. [6], there is a drastic difference between the models on the level of the fluctuations, provoked by the existence or nonexistence of an order parameter.

We recall shortly the model[6] and its properties which we need to establish the above results.

Let $\mathscr{H} = L^2(\mathbb{R}^1)$ and $\mathbb{Z}^d$ be the d-dimensional cubic lattice, $d > 2$. Let $\{Q_\ell, P_\ell\}$ be a copy of the canonical position and momentum operators Q and P on $\mathscr{H}$, i.e., $Q_\ell: \psi \in \mathscr{H}_\ell \to x_\ell \psi(x_\ell)$, $P_\ell: \psi \in \mathscr{H}_\ell \to (\hbar/i)\, \partial_{x_\ell} \psi(x_\ell)$ and $\mathscr{H}_\Lambda = \bigotimes_{\ell \in \Lambda} \mathscr{H}_\ell$. For each finite set (hypercube) $\Lambda \subset \mathbb{Z}^d$ the local Hamiltonian has the form

$$H_\Lambda(h) = \sum_{\ell \in \Lambda} \frac{P_\ell^2}{2m} + \frac{1}{4} \sum_{\ell, \ell' \in \Omega} \phi_{\ell\ell'} (Q_\ell - Q_{\ell'})^2 + \frac{a}{2} \sum_{\ell \in \Lambda} Q_\ell^2 + |\Lambda|\, W\left(\frac{1}{|\Lambda|} \sum_{\ell \in \Lambda} Q_\ell^2\right) - \sum_{\ell \in \Lambda} h Q_\ell \tag{4.1}$$

Here $\{\phi_{\ell\ell'}\}_{\ell, \ell' \in \mathbb{Z}^d}$ is the matrix of harmonic forces, $a > 0$ and the W-term mimics double-well one-site potentials responsible for the dispacive ferroelectric phases. For example $W(x) = \frac{1}{2} b \exp(-\eta x)$, $b > 0$, $\eta > 0$, see ref. [6]. The last term in (4.1) breaks the $Q_\ell \to -Q_\ell$ symmetry and is retained for the construction of different limiting (extremal) equilibrium states $\omega(\cdot)$.

Below the critical line $T_c(\lambda)$, $\lambda \in [0, \lambda_c)$, there are (at least) two translation invariant extremal equilibrium states $\omega_{\pm}(\cdot)$ defined by the quasi-average procedure:

$$\omega_{\pm}(\cdot) = \lim_{h \to \pm 0} \lim_{\Lambda} \omega_{\Lambda, h}(\cdot) \tag{4.2}$$

where $\omega_{\Lambda, h}$ is the state defined by the local Hamiltonian (4.1). These states correspond to the broken symmetry states with order parameter

$$\omega_{+}(Q_\ell) = -\omega_{-}(Q_\ell) \geqslant 0 \tag{4.3}$$

Moreover, straightforward calculations of the fluctuation operators in the extremal translation invariant states yield:[6]

$$\begin{aligned}
\omega(F^{\delta_Q}(Q)^2) &= \lim_{\Lambda} \omega_{\Lambda, h}(F^{\delta_Q}_{\Lambda}(Q)^2) \\
&= \lim_{\Lambda} \frac{1}{|\Lambda|^{2\delta_Q}} \frac{\lambda}{2\sqrt{\Delta(c_\Lambda)}} \operatorname{cth} \frac{1}{2} \beta\lambda \sqrt{\Delta(c_\Lambda)} \\
\omega(F^{\delta_P}(P)^2) &= \lim_{\Lambda} \omega_{\Lambda, h}(F^{\delta_P}_{\Lambda}(P)^2) \\
&= \lim_{\Lambda} \frac{1}{|\Lambda|^{2\delta_P}} \frac{\lambda m}{2} \sqrt{\Delta(c_\Lambda)} \operatorname{cth} \frac{1}{2} \beta\lambda \sqrt{\Delta(c_\Lambda)}
\end{aligned} \tag{4.4}$$

where $c_\Lambda \equiv c_\Lambda(h)$ verifies the equation

$$c_\Lambda = \frac{h^2}{\Delta^2(c_\Lambda)} + \frac{1}{|\Lambda|} \sum_{q \in \Lambda^*} \frac{\lambda}{2\Omega_q(c_\Lambda)} \operatorname{cth} \frac{1}{2} \beta\lambda\Omega_q(c_\Lambda) \tag{4.5}$$

Here Λ^* is dual of the hypercube Λ and $\Omega_q^2(c) = \Delta(c) + \omega_q^2$, where $\omega_q^2 = \tilde{\phi}(0) - \tilde{\phi}(q)$, $\tilde{\phi}(q)$ is the lattice Fourier-transform of the matrix $\phi_{\ell\ell'} = \phi_{\ell-\ell'}$, and $\Delta(c) = a + 2W'(c)$. There exists $c^* > 0$, such that $\Delta(c_\Lambda \geqslant c^*) \geqslant 0$, and $\Delta(c = c^*) = 0$.

The same type of calculations imply:

$$\omega_h(Q_\ell) = \lim_{\Lambda} \omega_{\Lambda, h}(Q_\ell) = \lim_{\Lambda} \frac{h}{\Delta(c_\Lambda(h))} \tag{4.6}$$

Proposition 4.1. Let $h_\Lambda = \hat{h}/|\Lambda|^\alpha$, then for $T > 0$ and below the critical line, for short-range interactions ($\omega_q^2 = s^2 q^2 + 0(q^2)$, $q \to 0$) one has:

(a) $\lim_{\Lambda} \omega_{\Lambda, h_\Lambda}(\cdot) = \omega_{\pm}(\cdot)$ if $0 < \alpha < 1$, here $\pm = \text{sign}\, \hat{h}$

(b) $\lim_{\Lambda} \omega_{\Lambda, h_\Lambda}(\cdot) = k\omega_{+}(\cdot) + (1-k)\,\omega_{-}(\cdot)$, $0 < k < 1$

a convex combination of extremal states, if $\alpha \geqslant 1$. □

For the proof we refer to [6, Proposition 4.2]. For us it will be important to remember that in the regime (a), the equation (4.5) below the critical line becomes:

$$0 < \rho(T, \lambda) = \lim_{\Lambda} \left(\frac{\hat{h}}{|\Lambda|^{\alpha} \Delta(c_\Lambda(h_\Lambda))} \right)^2 = c^* - I_d(c^*, T, \lambda) \tag{4.7}$$

where the integral I_d is given by:

$$I_d(c, T, \lambda) = \lim_{\Lambda} \frac{1}{|\Lambda|} \sum_{q \in \Lambda^*} \frac{\lambda}{2\Omega_q(c)} \operatorname{cth} \frac{1}{2} \beta\lambda\Omega_q(c) \tag{4.8}$$

Hence we get an extremal phase with the order parameter (4.6) equal to

$$\omega_{\pm}(Q_\ell) = (\text{sign}\, \hat{h}) \sqrt{\rho(T, \lambda)} \tag{4.9}$$

Corollary 4.2. The Proposition 4.1 is valid below the critical line for $T = 0$, i.e., for the interval $(0, \lambda_c)$, with the only modification of the range of α. The case (a) corresponds to $0 < \alpha < 2$, while the case (b) corresponds to $\alpha \geqslant 2$.

Proof. It is a straightforward consequence of the equation (4.5) modified for $T = 0$ and $0 < \lambda < \lambda_c$, that:

$$c_\Lambda = \left(\frac{\hat{h}}{|\Lambda|^{\alpha} \Delta(c_\Lambda)} \right)^2 + \frac{1}{|\Lambda|} \frac{\lambda}{2\sqrt{\Delta(c_\Lambda)}} + \frac{1}{|\Lambda|} \sum_{q \in \Lambda^*} \frac{\lambda}{2\Omega_q(c_\Lambda)} \tag{4.10}$$

For $0 < \alpha < 2$ one gets in the limit (cf. (4.7), (4.8)):

$$(\omega_{\pm}(Q_\ell))^2 = \lim_{\Lambda} \left(\frac{\hat{h}}{|\Lambda|^{\alpha} \Delta(c_\Lambda)} \right)^2 = c^* - I_d(c^*, T = 0, \lambda) \tag{4.11}$$

whereas for $\alpha = 2$, due to the $q = 0$ term in (4.10), we have

$$\omega_{-}(Q_\ell) < \lim_{\Lambda} \omega_{\Lambda, h_\Lambda}(Q_\ell) = \lim_{\Lambda} \frac{\hat{h}}{|\Lambda|^2 \Delta(c_\Lambda)} = <\omega_{+}(Q_\ell) \tag{4.12}$$

and for $\alpha > 2$ we obtain

$$\lim_{\Lambda} \omega_{\Lambda, h_\Lambda}(Q_\ell) = 0 \tag{4.13}$$

which corresponds to a mixed state, with $k = \frac{1}{2}$ (see Prop. 4.1(b)). □

Now we come back to the fluctuations (4.4) for pure states below the critical line.

Theorem 4.3. Let the breaking symmetry perturbation h_Λ in (4.1) be of the form $h_\Lambda = \hat{h}/|\Lambda|^\alpha$. Then for the short-range interaction below the critical line $T < T_c(\lambda)$, $0 < \lambda < \lambda_c$, in the extremal states $\omega_{\pm}(\cdot)$ we have:

$$\begin{aligned} &\text{(a)} \quad \delta_Q = \alpha/2, \quad \delta_P = 0, \quad \text{for } T > 0 \quad \text{and} \quad 0 < \alpha < 1; \\ &\text{(b)} \quad \delta_Q = -\delta_P = \alpha/4, \quad \text{for } T = 0 \quad \text{and} \quad 0 < \alpha < 2 \end{aligned} \tag{4.14}$$

Proof. (a) As far as below the critical line $\lim_\Lambda \Delta(c_\Lambda) = 0$, the case (a) results from (4.4) and (4.7).

(b) The same line of reasoning for (4.4) modified for $\beta = \infty$ (see (4.10)) gives immediately the result (4.14). □

Remark 4.4. On the critical line and for long-range interactions the fluctuations (4.4) are studied in refs. [8, 9, and 10]. For example, on the critical line at the point $T_c(\lambda_c) = 0$ one has $\delta_Q = -\delta_P$ where for short range interactions (see ref. [8], $d > 2$)

$$\delta_Q(\alpha) = \begin{cases} \alpha/6 & \text{for } 0 \leqslant \alpha < 1 \\ 1/6 & \text{for } 1 \leqslant \alpha \end{cases} \tag{4.15}$$

The case of mixed states below the critical line, $\omega = k\omega_+ + (1-k)\,\omega_-$, $0 < k < 1$, corresponds to $\alpha \geqslant 1$ in (4.14) (a) and $\alpha \geqslant 2$ in (4.14) (b).

Corollary 4.5. Non-abelian (quantum) nature of fluctuations: $\delta_Q + \delta_P = 0$, (4.4) *below the critical line* $T < T_c(\lambda)$ emerges only on the interval $(0, \lambda_c]$, $T = 0$. □

Hence, if the quantum parameter $\lambda > 0$ then the fluctuation operators $F^{\delta_Q}(Q)$, $F^{\delta_P}(P)$ in the pure ground states form a CCR-algebra (3.4). The same holds *above the critical line* while $\lambda = 0$, yields a classical algebra of fluctuations.

Let us note that results of the Theorem 4.3 are based on the scaling analysis of the spectral gap $\Delta(c_\Lambda) \sim |\Lambda|^{-\alpha}$ provoked by the symmetry breaking perturbation $h_\Lambda = \hat{h}/|\Lambda|^\alpha$, see (4.7) and (4.11).

Now we consider the fluctuation of Q^2, which yields together with P and Q, in the "toy harmonic model" a 4-dimensional Lie algebra of fluctuation operators for $\gamma \geqslant 1$.[3] The "toy harmonic model" is described in the following Hamiltonian:

$$H_A = \frac{1}{2m} \sum_{\ell \in A} P_\ell^2 + \frac{1}{4} \sum_{k, \ell \in A} \phi_{k\ell} (Q_k - Q_\ell)^2 + \frac{1}{2} \frac{g}{|A|^\gamma} \sum_{\ell \in A} Q_\ell^2; \quad \gamma > 0 \tag{4.16}$$

If one passes to the Fourier transforms for P_ℓ and Q_ℓ in (4.1) and (4.16), these Hamiltonians get the same formal expression except that the Fourier coordinates $Q_{(q)}$ are shifted in (4.1) to $\tilde{Q}_{(q)}$:

$$\tilde{Q}_{(q)} = Q_{(q)} - \frac{h_q}{\Omega_q^2} \tag{4.17}$$

where $h_q = h|A|^{1/2} \delta_{q,0}$, and the spectral gap $\Delta(c_A)$ is identified with $g/|A|^\gamma$.

For the model (4.1) we have:

Theorem 4.6. Let $F^{\delta_{Q^2}}(Q^2)$ be the fluctuation operator of the squares of the local displacements in the ordered pure $(0 \leqslant \alpha < 2)$ or mixed $(\alpha \geqslant 2)$ ground states perturbed by $h_A = \hat{h}/|A|^\alpha$. Then for $\lambda \in (0, \lambda_c)$ we have

$$\delta_{Q^2}(\alpha) = \begin{cases} \alpha/4 & \text{for} \quad 0 \leqslant \alpha < 2, \\ 1/2 & \text{for} \quad 2 \leqslant \alpha \end{cases} \tag{4.18}$$

whereas for $\lambda = \lambda_c$ one has $(d > 2)$

$$\delta_{Q^2}(\alpha) = 0 \qquad \text{for} \quad 0 \leqslant \alpha \tag{4.19}$$

The (4.19) is true for all $\lambda > \lambda_c$, as well as for the whole domain above the critical line $T_c(\lambda)$, i.e. in all disordered states.

Proof. Using the shift (4.17), the fluctuation operator $F_A^{\delta_{Q^2}}(Q^2)$ can be identically represented as the sum of the two fluctuations, namely as:

$$\begin{aligned} F_A^{\delta_{Q^2}}(Q^2) &= \frac{1}{|A|^{1/2+\delta_{Q^2}}} \sum_{\ell \in A} (Q_\ell^2 - \omega_A(Q_\ell^2)) \\ &= |A|^{\delta_{\tilde{Q}^2} - \delta_{Q^2}} F_A^{\delta_{\tilde{Q}^2}}(\tilde{Q}^2) + \frac{2\hat{h}}{|A|^{\delta_{Q^2} + \alpha - \delta_Q} \Delta(c_A)} F_A^{\delta_Q}(Q) \end{aligned} \tag{4.20}$$

This means that we have to study the asymptotic behaviour of two terms. In the pure ordered state, the behaviour of the gap $\Delta(c_\Lambda)$ is given by (see (4.7)):

$$\Delta(c_\Lambda) = \frac{\hbar}{|\Lambda|^\alpha \sqrt{\rho}}$$

Therefore the asymptotics of the coefficient of the second term in the representation (4.20) is given by $\simeq |\Lambda|^{\delta_Q - \delta_{Q^2}}$. The existence of the limiting fluctuation (4.20) is guaranteed by the existence of a non trivial limit for the coefficients. We know from [3] that for $\gamma = \alpha$:

$$\delta_{\tilde{Q}^2}(\alpha) = \begin{cases} 0 & \text{if} \quad 0 \leqslant \alpha < 1 \\ \dfrac{\alpha - 1}{2} & \text{if} \quad 1 \leqslant \alpha < 2 \end{cases} \tag{4.21}$$

On the other hand we know $\delta_Q(\alpha) = \alpha/4$ from Theorem 4.3(b).

Using this knowledge, in view of (4.20), we get immediately

$$\delta_{Q^2}(\alpha) = \max\{\delta_Q(\alpha), \delta_{\tilde{Q}^2}(\alpha)\} = \frac{\alpha}{4}$$

Now for $\lambda = \lambda_c$, from (4.4) one gets the asymptotics $\Delta(c_\Lambda) \simeq |\Lambda|^{-4\delta_Q}$. Then on the basis of the expression (4.20), one obtains the asymptotics of the coefficient of $F_\Lambda^{\delta_Q}(Q)$ given by $|\Lambda|^{\delta_{Q^2} + \alpha - 5\delta_Q}$. By (4.15): $\alpha - 5\delta_Q > 0$ for $\alpha > 0$. Again from ref. [3] for all values of $\gamma = 4\delta_Q > 0$ (4.16), we have $\delta_{\tilde{Q}^2}(\alpha) \geqslant 0$, therefore the coefficient of $F_\Lambda^{\delta_{Q^2}}(\tilde{Q}^2)$ can only be finite if and only if $\delta_{Q^2} \geqslant 0$. Hence $\delta_{Q^2} + \alpha - 5\delta_Q > 0$ and the second term in (4.20) always vanishes. Therefore $\delta_{Q^2} = \delta_{\tilde{Q}^2}$, and at the point λ_c, the models (4.1) and (4.16) coincide. For the toy model (4.16) we know that if the gap $\Delta(c_\Lambda)$ behaves like $|\Lambda|^{-\gamma}$, with $\gamma < 1$, then $\delta_{\tilde{Q}^2} = 0$ (cf. (4.21). Now due to (4.15): $\gamma = 4\delta_Q(\alpha) < \frac{2}{3} < 1$, hence $\delta_{Q^2}(\alpha) = 0$ for all $\alpha > 0$.

Finally if $\lambda > \lambda_c$, then the gap $\Delta(c_\Lambda)$ always stays strictly positive and all fluctuations remain normal. □

Now we want to clarify the difference in the quantum nature of the fluctuation operators in view of the ordered or disordered states. From the discussion in Section 3, it is clear that the most interesting situation in Theorem 3.2 is provided by the case (ii) of scale invariance: $\frac{1}{2} + \delta_j + \delta_k - \delta_\ell = 0$. In order to satisfy this equation in pure states (all δ's less than 1/2), one has to have that at least one of these exponents is negative. Therefore it is instructive to learn first what are the consequences of the presence of at least one squeezed operator in the triplet considered. In both of our models (4.1) as well as in (4.16), the operator P is the local operator with squeezed fluctuations on the line $T=0$ and $\lambda \in (0, \lambda_c]$.

Consider the following identity:

$$\lim_{\Lambda} |\Lambda|^{\delta_P + \delta_{Q^2}} \omega_\Lambda([F_\Lambda^{\delta_P}(P), F_\Lambda^{\delta_{Q^2}}(Q^2)]) = \lim_{\Lambda} 2 \frac{\hbar}{i} \omega_\Lambda(Q) \tag{4.22}$$

Remark that on the right hand side one gets the order parameter for the discrete symmetry ($Q \to -Q$) breaking in the anharmonic crystal model (4.1). By Theorem 4.3 and 4.6 one gets $\delta_P + \delta_{Q^2} = 0$ for all $\alpha : 0 \leqslant \alpha < 2$, which is in correspondence with $\lim_\Lambda \omega_\Lambda(Q) \neq 0$ in (4.22). On the other hand for the toy harmonic model (4.16), one has $\delta_P + \delta_{Q^2} < 0$ (4.14) and (4.21), which by (4.22) fits with the absence of order parameter or with $\lim_\Lambda \omega_\Lambda(Q) = 0$. This shows that our general result (Theorem 3.2) and the relative magnitude of the critical indices δ do give some information on the nature of the spontaneous symmetry breaking. Therefore it is natural to reconsider also the Goldstone theorem from our point of view. This theorem concerns systems in which a continuous symmetry group is spontaneously broken. A first attempt, preparing more general results, is given in the next section.

5. GOLDSTONE NORMAL MODE IN BOSE CONDENSATION

As an application of the general theorem above we give a mathematically rigorous derivation of the existence of the Goldstone normal mode in Bose condensation. It is popular wisdom in many-body theory as well as in field theory that each system, displaying order in some direction, i.e. showing spontaneous symmetry breaking, has collective modes which arise as a consequence of the broken symmetry. For short range interactions, the frequency of the collective mode tends to zero. The Goldstone theorem[12] predicts a sharp point $\omega = 0$ in the spectrum of the time evolution at $k = 0$.[13] In mathematical language,[14] the broken symmetry is not implementable. However up to now, there is not yet a general and mathematically rigorous construction of this Goldstone mode as a normal mode. Here we present this construction for the special case of the perfect Bose gas.

The situation of long range interactions is quite different. In this case one observes a frequency $\omega \neq 0$ at $k = 0$.[15–17] The mathematically rigorous setting of the corresponding mode has been settled in the one-component plasma[18] and in the Overhauser model.[19]

Coming back to the Bose gas, one considers a finite volume Λ in $\mathbb{R}^d$ $d \geqslant 3$ of size V, and the well known model

$$H_\Lambda = \sum_{k \in \Lambda^*} \omega_k a_k^* a_k + h \sqrt{V} (a_0^* + a_0) \tag{5.1}$$

where h is an external field which will tend to zero with the volume at the rate

$$h = \hat{h}/V^\alpha \qquad \alpha > 0 \tag{5.2}$$

Furthermore, the

$$a_k^* = \frac{1}{\sqrt{V}} \int_V dx\, e^{ikx} a^*(x); \qquad k \in \Lambda^*$$

are Boson creation operators, and $\omega_k = k^2/2m - \mu_\Lambda^h$, where μ_Λ^h is the chemical potential, determined by the density ρ, through the constraint:

$$\rho = \omega_\Lambda \left(\frac{N_\Lambda}{V} \right) = \frac{\operatorname{tr} e^{-\beta(H_\Lambda - \mu_\Lambda^h N_\Lambda)} N_\Lambda}{\operatorname{tr} e^{-\beta(H_\Lambda - \mu_\Lambda^h N_\Lambda)}}; \qquad N_\Lambda = \sum_k a_k^* a_k \tag{5.3}$$

for all volumes Λ. In the thermodynamic limit $\Lambda \to \mathbb{R}^d$, keeping the density ρ constant one gets, using (5.2):

$$\rho = \lim_\Lambda \left(\frac{\hat{h}}{\mu_\Lambda^h V^\alpha} \right)^2 + \lim_\Lambda \frac{1}{V} \frac{1}{e^{-\beta \mu_\Lambda^h} - 1} + \rho_c(\mu^0, T) \tag{5.4}$$

where

$$\rho_c(T) = I_q(T, 0) \quad \text{and} \quad \frac{1}{(2\pi)^d} \int dk \frac{1}{e^{\beta(\varepsilon_k - \mu^h)} - 1} \equiv I_d(T, \mu^h), \qquad \mu^0 = \lim_\Lambda \mu_\Lambda^h$$

One has breaking of the gauge symmetry in the zero-mode:

$$\lim_\Lambda \omega \left(\frac{a_0}{\sqrt{V}} \right)_\Lambda = \lim_\Lambda \frac{\hat{h}}{\mu_\Lambda^h V^\alpha}$$

The condensate density is given by

$$\rho_0 = \lim_\Lambda \omega \left(\frac{a_0^* a_0}{V} \right)_\Lambda = \lim_\Lambda \left(\frac{\hat{h}}{\mu_\Lambda^h V^\alpha} \right)^2 \tag{5.5}$$

We do not analyse the case $\alpha \geqslant 1$, because in that case, the limit Gibbs state, which one obtains is not an extremal invariant state, but a mixture of states with different phases, see ref. [20].

Now we consider the canonical coordinate of the zero-mode, normalized in the usual way:

$$q_\Lambda = \frac{a_0 + a_0^*}{\sqrt{2\omega_0}} = \frac{1}{\sqrt{2\omega_0}} \frac{1}{\sqrt{V}} \int_V dx(a(x) + a^*(x)) \tag{5.6}$$

This is except for the substracting of the average value, a local fluctuation of the operator $(a(x) + a^*(x))/\sqrt{2\omega_0}$, at the point x (see (1.2)). Our point is, that we give a meaning to the canonical coordinate of the zero-mode (5.6), by considering it as a fluctuation of the type considered in Section 2. In particular we consider

$$L_1(\Lambda) = \frac{1}{V^{1/2+\delta_1}} \int_\Lambda \frac{dx(a(x) + a^*(x) - \omega_\Lambda(a(x)) - \omega_\Lambda(a^*(x)))}{\sqrt{2\omega_0}} \tag{5.7}$$

First we look for the parameter δ_1 in order to satisfy Condition A (2.8). Compute:

$$\omega_\Lambda(L_1(\Lambda)^2) = \frac{1}{V^{2\delta_1}} \left(\frac{1}{2} + \frac{1}{e^{-\beta\mu_\Lambda^h} - 1}\right)\left(\frac{1}{-\mu_\Lambda^h}\right)$$

Using (5.5), Condition A (2.8) is satisfied if and only if

$$\delta_1 = \alpha \qquad \text{for} \quad T > 0 \tag{5.8}$$

$$\delta_1 = \frac{\alpha}{2} \qquad \text{for} \quad T = 0 \text{ (groundstate)} \tag{5.9}$$

By Lemma 2.1 this settles the limit of the canonical variable (5.7) with δ_1 given by (5.8) or (5.9):

$$\mathscr{L}_1 = \lim_\Lambda L_1(\Lambda) \tag{5.10}$$

Now the conjugate variable to (5.6) is the canonical momentum operator given by

$$p_\Lambda = i\sqrt{\frac{\omega_0}{2}}(a_0^* - a_0) = i[\omega_0 a_0^* a_0, q_\Lambda] \tag{5.11}$$

Again this leads us to consider the fluctuation of the local operator now given by $\sqrt{\omega_0/2}\,(a^*(x)-a(x))$. Consider now

$$L_2(\Lambda)=\frac{1}{V^{1/2+\delta_2}}\sqrt{\frac{\omega_0}{2}}\int_\Lambda dx(a^*(x)-a(x)-\omega_\Lambda(a^*(x))+\omega_\Lambda(a(x))) \quad (5.12)$$

Again we determine the parameter δ_2 by Condition A. Compute:

$$\omega_\Lambda(L_2(\Lambda)^2)=\frac{1}{V^{2\delta_2}}\left(\frac{-\mu_\Lambda^h}{2}\right)\left(\frac{1}{2}+\frac{1}{e^{-\beta\mu_\Lambda^h}-1}\right)$$

Using (5.5), one checks that (2.8) is satisfied if and only if

$$\delta_2=0 \qquad \text{for} \quad T>0 \quad (5.13)$$

$$\delta_2=-\frac{\alpha}{2} \qquad \text{for} \quad T=0 \text{ (groundstate)} \quad (5.14)$$

Again, following Section 2, this settles the limit of the momentum canonical variable (5.12) as a fluctuation operator, for all $T\geqslant 0$:

$$\mathscr{L}_2=\lim_\Lambda L_2(\Lambda) \quad (5.15)$$

Remark also that Condition B (2.10) is automatically satisfied, because we have only two generators L_1 and L_2.

The operators $\{\mathscr{L}_1, \mathscr{L}_2\}$ form the canonical pair of operators corresponding to the long wavelength (in fact $k=0$), low-frequency normal mode in the presence of the broken symmetry, namely the gauge symmetry.

It is interesting to remark that in the absence of condensation, these canonical variables are always normal fluctuations i.e. always $\delta_1=\delta_2=0$.

Now, applying Theorem 3.2, in the case of condensation, one gets the following Lie algebras:

(i) if $T>0$, then $\delta_1=\alpha>0$, see (5.8) and $\delta_2=0$, see (5.13), hence $\delta_1+\delta_2>0$, $\delta_0=\frac{1}{2}$ and therefore $[\mathscr{L}_1, \mathscr{L}_2]=0$ by Theorem 3.2(i). The Lie-algebra, generated by $\{\mathscr{L}_1, \mathscr{L}_2\}$ is abelian and the fluctuations behave classically.

(ii) if $T=0$, then $\delta_1=\alpha/2$, see (5.9) and $\delta_2=-\alpha/2$, see (5.14), hence $\delta_1+\delta_2=0$ and therefore: $[\mathscr{L}_1, \mathscr{L}_2]=-i\mathbb{1}$ Theorem 3.2(ii). We have bona fide quantum mechanical canonical commutation relations, with the phenomenon of squeezing of the momentum variable.

6. CONCLUSIONS

The results of this paper deal mainly with critical fluctuations. First of all fluctuations are defined as operators, which enables us to focus on the quantum effects seen at the level of fluctuations. The paper contains three main results. In Section 3 we prove a theorem about the Lie-algebra structure of fluctuations, based on the knowledge of the variances only. In Section 4, we discuss explicitly the quantum effect to detect the ground state of the toy harmonic crystal model and of an anharmonic crystal model. Apart for the illustrative aspects for our theorem, we want to point out that this computation reveals a rich fine structure of the fluctuation operator algebra. It is well known that below the critical temperature the model (4.1) has only two extremal equilibrium states. However at the level of the fluctuations, we find a whole class of states depending on the parameter α (see Theorem 4.3 and ref. [8] where the same phenomenon was discovered for critical fluctuations, i.e. degeneracy on the critical line). The latter parameter fixes the way the limit extremal state is prepared in our treatment. For classical spin systems this is done by the boundary conditions, for quantum systems boundary conditions are somewhat more mysterious.[23] However, the above phenomenon is not due to the specific quantum nature of the system but to the algebra of observables considered. In our case, the algebra of fluctuation operators, which is of course totally different from the algebra of local observables, enables us to detect this degeneracy.

In fact, this kind of phenomena was already observed in the one-dimensional Ising model.[24] There they considered a specific way of approaching the ground state ($T \to 0$) and revealed a non trivial structure of the set of observables related to the magnetisation fluctuation.

Finally in Section 5, we give an explicit mathematical coherent construction of the Goldstone normal mode which is a consequence of the spontaneous breaking of the gauge symmetry. Although, this result is only for the Bose gas, we are convinced that there is a general theorem behind, yielding a new point of view on the Goldstone theorem in any situation of spontaneous symmetry breaking. We will come back to this yet open problem, as well as to other aspects of the Goldstone theorem, in particular to the question of the Anderson's reconstruction theorem of symmetries.

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